NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

British Mathematical Olympiad

14th March, 1979

Time allowed - $3\frac{1}{2}$ hours

Write on one side of the paper only. Start each question on a fresh sheet of paper. Arrange your answers in order.

Put your full name, age (in years and months) and school on the top sheet of your answers. On each other sheet put your name and initials.

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1. Find all triangles ABC for which

$$AB + AC = 2 \text{ cm. and } AD + BC = \sqrt{5} \text{ cm.} ,$$

where AD is the altitude through A, meeting BC at right angles in D.

2. From a point O in 3-D space, three given rays OA, OB, OC emerge, the angles BOC, COA, AOB being α, β, γ respectively. $0 < \alpha, \beta, \gamma < \pi$.

Prove that, given $2s > 0$, there are unique points X, Y, Z on OA, OB, OC respectively such that the triangles YOZ, ZOX and XOY have the same perimeter $2s$, and express OX in terms of s and $\sin\frac{1}{2}\alpha$, $\sin\frac{1}{2}\beta$ and $\sin\frac{1}{2}\gamma$.

3. S is a set of distinct positive odd integers $\{a_i\}$, $i = 1$ to n. No two differences $|a_i - a_j|$ are equal, $1 \leqslant i < j \leqslant n$.

Prove $\sum_{i=1}^{n} a_i \geqslant \frac{1}{3} n(n^2 + 2)$.

4. The function f is defined on the rational numbers and takes only rational values. For all rational x and y

$$f(x + f(y)) = f(x)f(y).$$

Prove that f is constant.

5. For n a positive integer denote by $p(n)$ the number of ways of expressing n as the sum of one or more positive integers. Thus $p(4) = 5$, because there are 5 different sums, namely

1+1+1+1, 1+1+2, 1+3, 2+2, 4.

Prove that for $n > 1$,

$$p(n+1) - 2p(n) + p(n-1) \geqslant 0$$

6. Prove that in the infinite sequence of integers

10001, 100010001, 1000100010001, ...

there is no prime number.

Note each integer after the first (ten thousand and one) is obtained by adjoining 0001 to the digits of the previous integer.